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Note

Recurrence Formulas for Phases and Amplitudes of Spherical Functions of a Free Wave

In this work two-term recurrence formulas for the spherical Bessel's functions $j_L(\rho)$ and $n_L(\rho)$ are established, discussed and numerically applied.

1. Introduction

It is well known [2] that, with two linear operators defined in the interval $0 < \rho < +\infty$

$$h_\pm{}^L = (L/\rho) \pm (d/d\rho), \tag{1.1}$$

we can factor each of the radial equations for spherical functions of a free wave

$$\left[\frac{d^2}{d\rho^2} + \left(1 - \frac{L(L+1)}{\rho^2}\right)\right] u_L(\rho) = 0, \qquad L = 0, 1, 2,..., \tag{1.2}$$

as follows

$$h_-{}^L h_+{}^L u_L = u_L\,, \tag{1.3a}$$

$$h_+^{L+1} h_-^{L+1} u_L = u_L\,. \tag{1.3b}$$

On multiplying on the left (1.3a) by $h_+{}^L$ and (1.3b) by h_-^{L+1} one obtains, after comparing the results with (1.3b) and (1.3a), respectively,

$$u_{L-1} = h_+{}^L u_L \tag{1.4a}$$

and

$$u_{L+1} = h_-^{L+1} u_L\,. \tag{1.4b}$$

These are the basic recurrence formulas for the present work.

2. The Phases $\phi_L(\rho)$ and the Functions of the Amplitude $\zeta_L(\rho)$ and Their Asymptotic Behavior

Consider the spherical functions [5]

$$f_L(\rho) = \rho j_L(\rho), \qquad g_L(\rho) = \rho n_L(\rho) \tag{2.1}$$

which are two linearly independent solutions of radial Eq. (1.2).

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By definition

$$f_L = \frac{\sin \phi_L}{\zeta_L^{1/2}}, \qquad g_L = \frac{\cos \phi_L}{\zeta_L^{1/2}}. \tag{2.2}$$

Hence

$$\tan(\phi_L) = j_L/n_L \tag{2.3}$$

and

$$\zeta_L = 1/[\rho^2(j_L{}^2 + n_L{}^2)]. \tag{2.4}$$

Differentiating (2.3), using (2.4) and the Wronskian for the spherical Bessel's functions [5] one has

$$\zeta_L = d\phi_L/d\rho. \tag{2.5}$$

From the asymptotic behavior of $j_L(\rho)$ and $n_L(\rho)$ we deduce that for large values of ρ

$$\phi_{L(\rho)} \underset{\rho\to\infty}{\longrightarrow} \rho - (L\pi/2) \tag{2.6}$$

and, by differentiation

$$\lim_{\rho\to\infty} \zeta_L(\rho) = 1. \tag{2.7}$$

Note that the spherical functions in (1.4) may be written as

$$u_L(\rho) = af_L(\rho) + bg_L(\rho) \tag{2.8}$$

where a and b are constant coefficients.

3. Recurrence Relations for $\phi_L(\rho)$ and $\zeta_L(\rho)$

Choose

$$u_L(\rho) = g_L(\rho) + if_L(\rho). \tag{3.1}$$

According to (2.2) we may write

$$u_L(\rho) = \zeta_L^{-(1/2)} \exp(i\phi_L), \tag{3.2}$$

Substituting (3.2) in (1.4), we obtain, after putting $d\phi_L/d\rho = \zeta_L$,

$$\zeta_{L-1}^{-(1/2)} \cdot \exp(i\phi_{L-1}) = (\eta_L{}^+ + i\zeta_L)\, \zeta_L^{-(1/2)} \cdot \exp(i\phi_L), \tag{3.3a}$$

$$\zeta_{L+1}^{-(1/2)} \cdot \exp(i\phi_{L+1}) = (\eta_L{}^- - i\zeta_L)\, \zeta_L^{-(1/2)} \cdot \exp(i\phi_L), \tag{3.3b}$$

where

$$\eta_L{}^+ = -\frac{1}{2}\frac{1}{\zeta_L}\frac{d\zeta_L}{d\rho} + \frac{L}{\rho}$$

and

$$\eta_L^- = \frac{1}{2}\frac{1}{\zeta_L}\frac{d\zeta_L}{d\rho} + \frac{L+1}{\rho}.$$

We immediately see that

$$\eta_L^+ + \eta_L^- = (2L+1)/\rho. \tag{3.4}$$

Separating real and imaginary parts and using logarithms we easily obtain for (3.3a)

$$\phi_{L-1} = \phi_L - \tan^{-1}\left(\frac{\eta_L^+}{\zeta_L}\right) + \frac{\pi}{2}, \tag{3.5a}$$

$$\zeta_{L-1} = \frac{1}{(\eta_L^+)^2 + (\zeta_L)^2}\,\zeta_L, \tag{3.5b}$$

and, for (3.3b),

$$\phi_{L+1} = \phi_L + \tan^{-1}\left(\frac{\eta_L^-}{\zeta_L}\right) - \frac{\pi}{2}, \tag{3.6a}$$

$$\zeta_{L+1} = \frac{1}{(\eta_L^-)^2 + (\zeta_L)^2}\cdot\zeta_L. \tag{3.6b}$$

Change L into $L-1$ in (3.6a) and compare the result with (3.5a). We obtain

$$\frac{\eta_{L-1}^-}{\zeta_{L-1}} = \frac{\eta_L^+}{\zeta_L}. \tag{3.7}$$

Therefore, if we change L into $L-1$ in (3.4) and substitute η_{L-1}^- from (3.7), we obtain

$$\eta_{L-1}^+ = \frac{2L-1}{\rho} - \frac{\zeta_{L-1}}{\zeta_L}\,\eta_L^+. \tag{3.5c}$$

Similarly

$$\eta_{L+1}^- = \frac{2L+3}{\rho} - \frac{\zeta_{L+1}}{\zeta_L}\,\eta_L^-. \tag{3.6c}$$

Equations (3.5) and (3.6) give the values of $\phi_L(\rho)$, $\zeta_L(\rho)$, and $\eta_L^-(\rho)$ (or $\eta_L^+(\rho)$) for any L, once the same functions are known for a particular value of L.

Using (3.5b), (3.6b), and (2.7) we have

$$\lim_{\rho\to\infty} \eta_L^+(\rho) = \lim_{\rho\to\infty} \eta_L^-(\rho) = 0. \tag{3.8}$$

The set of recurrence formulas (3.6) have already been obtained for spherical Coulomb functions [3, 4].

4. $\zeta_L(\rho)$ AND $\eta_L^{\pm}(\rho)$ AS RATIONAL FUNCTIONS OF ρ

To obtain these functions we require the explicit forms for the $f_L(\rho)$ and $g_L(\rho)$, or for the $u_L(\rho)$ as defined in (3.1). It is well known [5] that

$$\begin{aligned} u_L &= g_L + if_L \\ &= \exp\left[i\left(\rho - \frac{L\pi}{2}\right)\right](C_L + iS_L), \end{aligned} \tag{4.1}$$

where

$$C_L + iS_L = \sum_{n=0}^{L} \frac{(L+n)!}{n!(L-n)!}\left(\frac{i}{2\rho}\right)^n. \tag{4.2}$$

A simple inspection of (4.1) and (4.2) shows that

$$[g_L(\rho) + if_L(\rho)]_{\rho\to\infty} \to \exp[i(\rho - (L\pi/2))]. \tag{4.3}$$

Thus, the above formulas for $f_L(\rho)$ and $g_L(\rho)$ have correct asymptotic behavior.

From (2.1), (2.4), and (3.4) we now obtain

$$\zeta_L = 1/[(C_L)^2 + (S_L)^2], \tag{4.4}$$

$$\eta_L^+ = \left(C_L \frac{dC_L}{d\rho} + S_L \frac{dS_L}{d\rho}\right)\Big/[(C_L)^2 + (S_L)^2] + L/\rho, \tag{4.5a}$$

$$\eta_L^- = -\left(C_L \frac{dC_L}{d\rho} + S_L \frac{dS_L}{d\rho}\right)\Big/[(C_L)^2 + (S_L)^2] + (L+1)/\rho, \tag{4.5b}$$

In Table I we give the analytical expressions of $\phi_L(\rho)$, $\zeta_L(\rho)$, and $\eta_L^-(\rho)$ for $L=0, 1, 2$.

TABLE I

	$L = 0$	$L = 1$	$L = 2$
$\phi_L(\rho)$	ρ	$\rho + \tan^{-1}(1/\rho) - (\pi/2)$	$\rho + \tan^{-1}\left(\frac{1}{\rho}\right) + \tan^{-1}\left(\frac{3+2\rho^2}{\rho^3}\right) - \pi$
$\zeta_L(\rho)$	1	$\rho^2/(1+\rho^2)$	$[\rho^4(1+\rho^2)]/[(3+2\rho^2)^2+\rho^6]$
$\eta_L^-(\rho)$	$1/\rho$	$(3+2\rho^2)/[\rho(1+\rho^2)]$	$5/\rho - [\rho(1+\rho^2)(3+2\rho^2)]/[(3+2\rho^2)^2+\rho^6]$

The $\zeta_L(\rho)$ and $\eta_L^-(\rho)$ are obtained directly from (4.4) and (4.5b). The $\phi_L(\rho)$ are derived from (3.6a) taking $\phi_0 = \rho$. In fact, from (2.1), (4.1), and (4.2) we have

$$j_0 = \sin\rho/\rho, \qquad n_0 = \cos\rho/\rho \tag{4.6}$$

and from (2.3), we obtain

$$\tan(\phi_0) = \tan\rho, \tag{4.7}$$

5. Recurrence Relations for $f_L(\rho)$ and $g_L(\rho)$ or $j_L(\rho)$ and $n_L(\rho)$

Consider Eqs. (3.1), (3.3a), and (3.3b). We have

$$g_{L-1} + if_{L-1} = (\eta_L^+ + i\zeta_L)(g_L + if_L), \tag{5.1a}$$

$$g_{L+1} + if_{L+1} = (\eta_L^- - i\zeta_L)(g_L + if_L), \tag{5.1b}$$

or, by separating the real and imaginary parts,

$$f_{L-1} = \eta_L^+ f_L + \zeta_L g_L \,, \tag{5.2a}$$

$$g_{L-1} = \eta_L^+ g_L - \zeta_L f_L \,, \tag{5.2b}$$

and, similarly,

$$f_{L+1} = \eta_L^- f_L - \zeta_L g_L \,, \tag{5.3a}$$

$$g_{L+1} = \eta_L^- g_L + \zeta_L f_L \,. \tag{5.3b}$$

Use (2.1) to produce two-term recursion formulas for $j_L(\rho)$ and $n_L(\rho)$

$$\begin{aligned} j_0 &= \sin\rho/\rho, \\ n_0 &= \cos\rho/\rho, \\ \zeta_0 &= 1, \\ \eta_0^- &= 1/\rho, \end{aligned} \tag{5.4}$$

$$\begin{aligned} j_{L+1} &= \eta_L^- j_L - \zeta_L n_L \,, \\ n_{L+1} &= \eta_L^- n_L + \zeta_L j_L \,, \\ \zeta_{L+1} &= \zeta_L/[(\eta_L^-)^2 + (\zeta_L)^2], \\ \eta_{L+1}^- &= \frac{2L+3}{\rho} - \frac{\zeta_{L+1}}{\zeta_L}\,\eta_L^- . \end{aligned} \tag{5.5}$$

6. Behavior of $\phi_L(\rho)$, $\zeta_L(\rho)$, $\eta_L^{\pm}(\rho)$, and the Spherical Functions at the Origin and for $L \gg \rho$

The following discussion is necessary for the analysis of the numerical results given in the next section.

From (4.2) and (4.4) we have when ρ is small

$$\zeta_L(\rho)_{\rho\to 0} \to \left[\frac{2L+1}{(2L+1)!!}\right]^2 \rho^{2L} \left[1 - \frac{\rho^2}{2L-1} + \cdots\right] \tag{6.1}$$

and, by integrating (see (2.5)) $\zeta_L(\rho)$ from zero to ρ,

$$\phi_L(\rho)_{\rho\to 0} \to \frac{2L+1}{[(2L+1)!!]^2}\,\rho^{2L+1} \left[1 - \frac{(2L+1)\rho^2}{(2L+3)(2L-1)} + \cdots\right]. \tag{6.2}$$

We remark that the lower limit of integration has been taken equal to zero because $f_L(0) = 0$ requires that $\phi_L(0) = 0$ (see (2.1)).

Next, from (2.2), (6.1), and (6.2) we obtain

$$f_L(\rho)_{\rho\to 0} \to \frac{\rho^{L+1}}{(2L+1)!!}\left[1 - \frac{\rho}{2(2L+3)} + \cdots\right] \tag{6.3a}$$

and

$$g_L(\rho)_{\rho\to 0} \to \frac{(2L+1)!!}{2L+1}\left(\frac{1}{\rho}\right)^L \left[1 + \frac{\rho^2}{2(2L-1)} + \cdots\right]. \tag{6.3b}$$

Next, we shall prove that the first two terms of the developments at the origin of $\zeta_L(\rho)$, $\phi_L(\rho)$, $f_L(\rho)$ and $g_L(\rho)$ shown above are exactly the same as those of the same functions for $L \gg \rho$, so that, maintaining ρ fixed, we may substitute $\rho \to 0$ by $L \to \infty$ in all the formulas (6.1), (6.2), and (6.3).

Suppose, then, that $L \gg \rho$. Radial Eq. (1.2) can be written approximately in this case as

$$\frac{d^2u_L}{d\rho^2} - \frac{L(L+1)}{\rho^2}u_L \simeq 0.$$

This equation has two linearly independent solutions ρ^{L+1} and $(1/\rho)^L$. Make the variable transformation $u_L = \rho^{L+1}v_L$ in radial Eq. (2.1). We have

$$\frac{d^2v_L}{d\rho^2} + \frac{2(L+1)}{\rho}\frac{dv_L}{d\rho} + v_L = 0. \tag{6.4}$$

This equation for $v_L(\rho)$ can be solved approximately, in the case under consideration of $L \gg \rho$, by neglecting the term in $d^2v_L/d\rho^2$. We have then

$$v_L \approx c_L \exp\left[-\frac{\rho^2}{4(L+1)}\right].$$

Putting this solution back into the exact Eq. (6.4) we find that the error made in this approximation is equal to $\{[\rho/(2L+2)]^2 - 1/(2L+2)\}v_L$. Comparing the solution of $u_L(\rho)$ obtained in this way with the development (6.3a) of $f_L(\rho)$, we see that, apart from a multiplicative constant c_L, they agree in the first two terms, except for a slight difference in the denominator of the argument of the exponential. Therefore, if we correct the function for $v_L(\rho)$ as follows

$$v_L(\rho) \simeq c_L \exp\left[-\frac{\rho^2}{2(2L+3)}\right], \tag{6.5}$$

and put $c_L = 1/(2L+1)!!$, we obtain for $f_L(\rho)$

$$f_L(\rho)_{\rho\to 0,\ \mathrm{or}\ L\to\infty} \to \frac{\rho^L}{(2L+1)!!}\exp\left[-\frac{\rho^2}{2(2L+3)}\right], \tag{6.6a}$$

where the symbol ($\rho \to 0$, or $L \to \infty$) means that the statements $\rho \to 0$ and $L \to \infty$ cannot be valid simultaneously. Evidently, the error made in (6.4) by using the function (6.5) for $v_L(\rho)$ is reduced to $[\rho/(2L+3)]^2 v_L$.

Similarly, by making the variable transformation $u_L(\rho) = v_L(\rho)/\rho^L$, corresponding to the solution $(1/\rho)^L$ of the approximate radial equation, and by following the same steps we just took for obtaining the two-limit formula (6.6a) for $f_L(\rho)$,

$$g_L(\rho)_{\rho\to 0,\text{ or } L\to\infty} \to \frac{(2L+1)!!}{2L+1}\left(\frac{1}{\rho}\right)^L \exp\left[\frac{\rho^2}{2(2L-1)}\right]. \tag{6.6b}$$

Therefore, by (2.3)

$$\phi_L(\rho)_{\rho\to 0,\text{ or } L\to\infty} \to \frac{(2L+1)\rho^{2L+1}}{[(2L+1)!!]^2}\exp\{-\rho^2(2L+1)/[(2L+3)(2L-1)]\} \tag{6.7}$$

and, by differentiation (see (2.5))

$$\zeta_L(\rho)_{\rho\to 0,\text{ or } L\to\infty} \to \left[\frac{2L+1}{(2L+1)!!}\right]^2 \rho^{2L}\exp\left(-\frac{\rho^2}{2L-1}\right). \tag{6.8}$$

Note that the first two terms in the developments of (6.7) and (6.8) are exactly the same as those of the developments given respectively, in (6.2) and (6.1) for $\phi_L(\rho)$ and $\zeta_L(\rho)$.

Next, we calculate $\frac{1}{2}(d/d\rho)(\log \zeta_L)$ from (6.8) and by introducing this term into the definitions of $\eta_L^{\pm}(\rho)$ (see formulas above (3.4)) we determine how these functions behave at the origin or for $L \gg \rho$

$$\eta_L^{+}(\rho)_{\rho\to 0,\text{ or } L\to\infty} \to \frac{\rho}{2L-1} \tag{6.9a}$$

and

$$\eta_L^{-}(\rho)_{\rho\to 0,\text{ or } L\to\infty} \to \frac{2L+1}{\rho} - \frac{\rho}{2L-1}. \tag{6.9b}$$

Before closing this section we would like to point out that, according to radial Eq. (1.2), any of its solutions has points of inflexion at $\rho_0 = (L(L+1))^{1/2}$ or at any one of its zeros. Therefore, the solutions $f_L(\rho)$ and $g_L(\rho)$ of this equation, that are positive near the origin (see (6.3)) and have positive curvatures there, cannot change signs before crossing a point of inflexion. As their zeros are beyond ρ_0, with the exception of the origin belonging to $f_L(\rho)$ (see [1, p. 440]), we can say that $f_L(\rho)$ and $g_L(\rho)$ (and consequently $j_L(\rho)$ and $n_L(\rho)$) are defined positive functions in the interval $0 \leqslant \rho \leqslant \rho_0$.

It is also well known that the zeros of $f_L(\rho)$ and $g_L(\rho)$ interlace. Therefore the first zero of $g_L(\rho)$ lies between ρ_0 and the second zero of $f_L(\rho)$.

This property helps to understand the behavior of the phases with ρ. In fact $\phi_L(\rho)$ is an increasing positive function on the interval $0 \leqslant \rho \leqslant +\infty$, because it is positive near the origin (see (6.2) or (6.7)) and its derivative $\zeta_L(\rho)$ is also a positive function in the same interval (see Section 4). Therefore, when ρ goes from zero to infinity, $\phi_L(\rho)$ starts at the origin, then increases up to $\pi/2$ when ρ attains the first zero of $g_L(\rho)$, then up to π when ρ reaches the second zero of $f_L(\rho)$, then goes into $3\pi/2$ at the second zero of $g_L(\rho)$ and so on.

7. Numerical Calculations. Instability of Some of the Preceding Functions

Tables II and III give some numerical values of the functions $\phi_L(\rho)$, $j_L(\rho)$, and $n_L(\rho)$ for $\rho = 10$.

The columns marked "FORWARD" were calculated using recurrence procedures that start at $L = 0$ and go up step by step until $L = 30$. The columns marked "BACKWARD" were started at $L = 30$ and carried on by steps of one unit for a decreasing L until $L = 0$.

TABLE II

The "FORWARD" Column is Unstable below the Arrow.
The "BACKWARD" Column is Stable for any L
$\rho = 10$

L	$\phi_L(\rho)$ FORWARD	$\phi_L(\rho)$ BACKWARD
0	.10000000E 02	.10000000E 02
1	.85288723E 01	.85288723E 01
2	.71583545E 01	.71583545E 01
3	.58903975E 01	.58903975E 01
4	.47281083E 01	.47281083E 01
5	.36759911E 01	.36759911E 01
6	.27402982E 01	.27402982E 01
7	.19294626E 01	.19294626E 01
8	.12543731E 01	.12543731E 01
9	.72753693E 00	.72753693E 00
10	.35844032E 00	.35844032E 00
11	.14149005E 00	.14149005E 00
12	.42803519E-01	.42803519E-01
13	.98857258E-02	.98857258E-02
14	.17966495E-02	.17966495E-02
15	.26641372E-03	.26641372E-03
16	.33142982E-04	.33142982E-04
17	.35281110E-05	.35281110E-05
18	.32610769E-06	.32610769E-06
→19	.26475452E-07	.26475452E-07
20	.19058388E-08	.19058386E-08
21	.12261503E-09	.12261475E-09
22	.70992101E-11	.70987313E-11
23	.37259085E-12	.37203805E-12
24	.18429702E-13	.17743705E-13
25	.15543122E-14	.77371850E-15
26	.88817842E-15	.30976070E-16
27	.88817842E-15	.11429347E-17
28	.88817842E-15	.39000057E-19
29	.88817842E-15	.12345976E-20
30	.88817842E-15	.36362789E-22

TABLE III

The "FORWARD" Column for $j_L(\rho)$ is Unstable below the Arrow. The Other Two Columns are Stable for any L

$\rho = 10$

L	$j_L(\rho)$ FORWARD	$j_L(\rho)$ BACKWARD	$n_L(\rho)$ FORWARD
0	-.54402111E-01	-.54402111E-01	-.83907153E-01
1	.78466942E-01	.78466942E-01	-.62792826E-01
2	.77942194E-01	.77942194E-01	.65069305E-01
3	-.39495845E-01	-.39495845E-01	.95327479E-01
4	-.10558929E 00	-.10558929E 00	.16599302E-02
5	-.55534512E-01	-.55534512E-01	-.93833542E-01
6	.44501322E-01	.44501322E-01	-.10487683E 00
7	.11338623E 00	.11338623E 00	-.42506332E-01
8	.12557802E 00	.12557802E 00	.41117328E-01
9	.10009641E 00	.10009641E 00	.11240579E 00
10	.64605154E-01	.64605154E-01	.17245367E 00
11	.35574415E-01	.35574415E-01	.24974692E 00
12	.17216000E-01	.17216000E-01	.40196425E 00
13	.74655845E-02	.74655845E-02	.75516370E 00
14	.29410783E-02	.29410783E-02	.16369777E 01
15	.10635427E-02	.10635427E-02	.39920717E 01
16	.35590407E-03	.35590407E-03	.10738445E 02
17	.11094073E-03	.11094073E-03	.31444796E 02
→18	.32388474E-04	.32388474E-04	.99318340E 02
19	.88966268E-05	.88966273E-05	.33603306E 03
20	.23083702E-05	.23083720E-05	.12112106E 04
21	.56769095E-06	.56769777E-06	.46299304E 04
22	.13270091E-06	.13272846E-06	.18697490E 05
23	.29463159E-07	.29580290E-07	.79508775E 05
24	.57759327E-08	.62989045E-08	.35499375E 06
25	-.11610885E-08	.12843422E-08	.16599606E 07
26	-.11697484E-07	.25124088E-09	.81108054E 07
27	-.60835578E-07	.47234414E-10	.41327308E 08
28	-.32289819E-06	.85483986E-11	.21918939E 09
29	-.17796841E-05	.14914584E-11	.12080522E 10
30	-.10177238E-04	.25120574E-12	.69083186E 10

Note that only the "backward" recurring procedure is stable for $\phi_L(\rho)$ and $j_L(\rho)$. On the contrary the "forward" recurring procedure is stable for $n_L(\rho)$. (Consult [1, p. 452; 6]).

In Tables II and III the arrows indicate the values of $L(>\rho)$ up to which we can go with a precision of eight correct figures for $\phi_L(\rho)$ and $j_L(\rho)$ by using the corresponding "forward" recurrence formulas in a double-precision FORTRAN IV (15 digits) programme.

Consider the mechanism of the instability of $\phi_L(\rho)$ and $f_L(\rho)$ (or $j_L(\rho)$) when L becomes larger than ρ in a "forward" (increasing L) recurrence procedure.

Take, for instance, $\phi_L(\rho)$. We can rewrite (3.6a) as

$$\phi_{L+1}(\rho) = \phi_L(\rho) - \tan^{-1}\left[\frac{\zeta_L(\rho)}{\eta_L^-(\rho)}\right]. \tag{7.1}$$

Now, from (6.7) and (6.8) we have for $L \gg \rho$

$$\phi_L(\rho) \simeq \frac{\rho}{2L+1}\cdot\zeta_L(\rho)\exp\left[\frac{2\rho^2}{(2L+3)(2L-1)}\right]. \tag{7.2}$$

Similarly we obtain from (6.9b) for the same L

$$\eta_L^-(\rho) \simeq \frac{2L+1}{\rho}\exp\left[-\frac{\rho^2}{(2L+1)(2L-1)}\right]. \tag{7.3}$$

Thus we have approximately for the second term in the recurrence relation (7.1)

$$\tan^{-1}\left[\frac{\zeta_L(\rho)}{\eta_L^-(\rho)}\right] \simeq \frac{\rho}{2L+1}\,\zeta_L(\rho)\exp\left[\frac{\rho^2}{(2L+1)(2L-1)}\right], \tag{7.4}$$

i.e., a term of the same sign as $\phi_L(\rho)$ in (7.2) and very close to it in magnitude. Hence, the difference ϕ_{L+1} between these terms in (7.1) will be unstable for large L.

Next we consider $f_L(\rho)$ or $j_L(\rho) = f_L(\rho)/\rho$. The two terms in the recurrence formula (5.3a), i.e.,

$$f_{L+1}(\rho) = \eta_L^-(\rho)\,f_L(\rho) - \zeta_L(\rho)\,g_L(\rho) \tag{5.3a}$$

give approximately for $L \gg \rho$ (see 7.2) and (7.3))

$$\eta_L^-(\rho)\,f_L(\rho) \simeq \zeta_L^{1/2}(\rho)\exp\left[\frac{\rho^2}{(2L+3)(2L+1)}\right] \tag{7.5}$$

and

$$\zeta_L(\rho)\,g_L(\rho) \simeq \zeta_L^{1/2}(\rho). \tag{7.6}$$

Again these two terms have the same sign and are very close to one another in magnitude, so that their difference ($f_{L+1}(\rho)$) becomes numerically inaccurate with increasing L.

Finally we deal with the problem of calculating the initial values to start the "backward" (decreasing L) recurring procedure.

The fundamental relation for such a calculation is the recurrence formula (3.5a) rewritten as

$$\phi_L(\rho) = \phi_{L+1}(\rho) + \tan^{-1}\left[\frac{\zeta_{L+1}(\rho)}{\eta_{L+1}^+(\rho)}\right] \tag{7.7}$$

from which we derive the rapidly convergent series

$$\phi_L(\rho) = \sum_{s=1}^{\infty}\tan^{-1}\left[\frac{\zeta_{L+s}(\rho)}{\eta_{L+s}^+(\rho)}\right], \tag{7.8a}$$

when $L > \rho$ (see (6.8) and (6.9a)).

If L is sufficiently larger than ρ, so that $\tan^{-1}[\zeta_L(\rho)/\eta_L^+(\rho)] \simeq \zeta_L(\rho)/\eta_L^+(\rho)$ is a good approximation, we can simplify (7.8a) and write

$$\phi_L(\rho) \simeq \sum_{s=1}^{N} \frac{\zeta_{L+s}(\rho)}{\eta_{L+s}^+(\rho)}. \tag{7.8b}$$

For such values of L and from (5.2b), (6.5), and (6.6) we also have the approximate relations

$$g_L(\rho) \simeq \eta_{L+1}^+(\rho)\, g_{L+1}(\rho) \tag{7.9}$$

and

$$\zeta_L(\rho) \simeq 1/[g_L(\rho)]^2. \tag{7.10}$$

Therefore we can write (7.8b) as

$$\phi_L(\rho) \simeq \sum_{s=0}^{N} \frac{1}{g_{L+s}(\rho)\, g_{L+s+1}(\rho)} \tag{7.11}$$

and from (2.1) and (2.2)

$$f_L(\rho) \simeq g_L(\rho) \sum_{s=0}^{N} \frac{1}{g_{L+s}(\rho)\, g_{L+s+1}(\rho)}. \tag{7.12}$$

The remarkable thing about the expansion (7.12) is that it can be obtained exactly (see [6]) from the Wronskian relation

$$f_L(\rho)\, g_{L+1}(\rho) - f_{L+1}(\rho)\, g_L(\rho) = 1, \tag{7.13a}$$

written as

$$\frac{f_L(\rho)}{g_L(\rho)} = \frac{f_{L+1}(\rho)}{g_{L+1}(\rho)} + \frac{1}{g_L(\rho)\, g_{L+1}(\rho)}. \tag{7.13b}$$

From this recurrence formula for the ratio $f_L(\rho)/g_L(\rho)$ we immediately obtain (7.12). We note that, according to the discussion at the end of Section 6, the $g_L(\rho)$ cannot vanish for $L > \rho$. Thus, the terms of the development (7.12) for $f_L(\rho)$ never become infinite.

The functions $\eta_L^+(\rho)$, required in the "backward" recurrence procedure, must be calculated by means of the formula (see (3.6b) and (3.7)).

$$\eta_{L+1}^+ = \eta_L^-/[(\eta_L^-)^2 + (\zeta_L)^2]. \tag{7.14}$$

The "backward" recurrence relation (3.5c) for the determination of the $\eta_L^+(\rho)$ is numerically unstable for $L \gg \rho$. This can be seen by following similar steps as those taken in this section in showing the numerical instability of the recurrence relations (3.6a) and (5.3a) for $\phi_L(\rho)$ and $f_L(\rho)$, respectively.

Finally we would like to point out that the "foreward" (increasing L) recurrence formulas presented in this article for $j_L(\rho)$ and $f_L(\rho)$ are more stable than the familiar three-term recurrence relation [1, 6].

All the calculations were performed at the Coimbra University with the SIGMA 5 XEROX computer.

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